

Derivation of the 6DOF ECEF equations of motion

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Nomenclature

$[\mathbf{C}]_{b/a}$ - Direction cosine matrix transformation from coordinate system a to b

$()^a$ - Vector in coordinate system a

${}^b\dot{()}^a$ - Derivative of vector with respect to coordinate system b expressed in coordinate system a

$()_{b/a}$ - b with respect to a

μ - longitude

l - latitude

h - altitude

Re - Earth radius

Derivation of the equations of motion

The flat-Earth equations of motion are used for short-distance flight only and for observation of the aircraft's response to control inputs. Here the aircraft equations of motion for spherical, rotating Earth are derived. These equations are widely used for aircraft simulation.

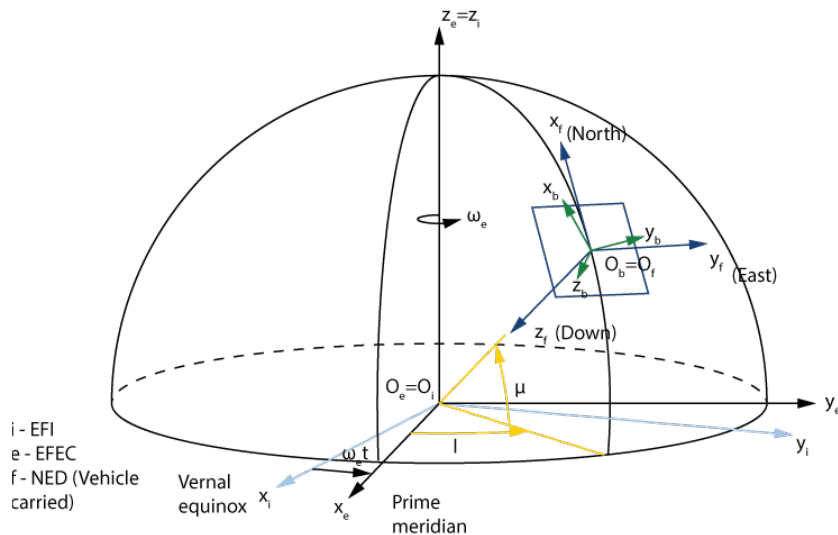


Figure 1: EFEC

To derive the equations of motion around spherical, rotating Earth, four coordinate systems are defined. The Earth-centred inertial (ECI) coordinate system has its origin at the centre of the Earth with its x_i -axis pointing towards the vernal equinox and its z_i -axis pointing towards the Earth's north pole. The Earth-fixed Earth-centred coordinate system (EFEC) has its origin at the centre of the Earth with its x_e -axis pointing towards the prime meridian and z_e -axis towards the Earth's north pole. The north-east-down (NED) coordinate system has its origin at a particular longitude, latitude and altitude with respect to the EFEC coordinate system, its x_f -axis is pointing north and its z_f -axis is pointing towards the centre of the Earth. The last coordinate system is the body-fixed coordinate system. Its origin coincides with the origin of the NED coordinate system origin.

The direction cosine matrices between the EFEC, NED, and body-fixed coordinate systems must be defined. The direction cosine matrix from the EFEC to NED coordinate system in terms of the longitude (μ) and latitude (l) is:

$$[\mathbf{C}]_{f/e} = \begin{bmatrix} -C_l S_\mu & -S_l S_\mu & C_\mu \\ -S_l & C_l & 0 \\ -C_l C_\mu & -C_\mu S_l & -S_\mu \end{bmatrix}, \quad (1)$$

and the direction cosine matrix from the NED to the body-fixed coordinate system in terms of the Euler angles (ϕ, θ, ψ) is:

$$[\mathbf{C}]_{b/f} = \begin{bmatrix} C_\theta C_\psi & C_\theta S_\psi & -S_\theta \\ S_\phi S_\theta C_\psi - C_\phi S_\psi & S_\phi S_\theta S_\psi + C_\phi C_\psi & S_\phi C_\theta \\ C_\phi S_\theta C_\psi + S_\phi S_\psi & C_\phi S_\theta S_\psi - S_\phi C_\psi & C_\phi C_\theta \end{bmatrix}. \quad (2)$$

The direction cosine matrix from the ECEF to the body-fixed coordinate system is:

$$[\mathbf{C}]_{b/e} = [\mathbf{C}]_{f/e} [\mathbf{C}]_{b/f}. \quad (3)$$

A position vector \mathbf{r}^e in the ECEF coordinate system gives the position of the NED and the body-fixed coordinate systems (the NED and body-fixed coordinate systems coincide, as mentioned earlier). The time derivative of \mathbf{r}^e with respect to the ECI coordinate system is given by:

$${}^i \dot{\mathbf{r}}^e = {}^e \dot{\mathbf{r}}^e + \omega_{e/i}^e \times \mathbf{r}^e, \quad (4)$$

where $\omega_{e/i}^e$ is the earth rotational velocity vector in the ECEF frame. Differentiating the above equation again gives:

$${}^i \ddot{\mathbf{r}}^e = {}^e \ddot{\mathbf{r}}^e + {}^e \dot{\omega}_{e/i}^e \times \mathbf{r}^e + \omega_{e/i}^e \times {}^e \dot{\mathbf{r}}^e + \omega_{e/i}^e \times \omega_{e/i}^e \times \mathbf{r}^e + \omega_{e/i}^e \times \omega_{e/i}^e \times \mathbf{r}^e. \quad (5)$$

Simplifying the above equation by gathering terms gives:

$${}^i \ddot{\mathbf{r}}^e = {}^e \ddot{\mathbf{r}}^e + {}^e \dot{\omega}_{e/i}^e \times \mathbf{r}^e + 2\omega_{e/i}^e \times {}^e \dot{\mathbf{r}}^e + \omega_{e/i}^e \times \omega_{e/i}^e \times \mathbf{r}^e. \quad (6)$$

Using Newton's Second Law (often stated as $F = ma$) gives:

$${}^i \ddot{\mathbf{r}}^e = \mathbf{a}^e = \frac{\mathbf{F}^e}{m} + \mathbf{G}^e, \quad (7)$$

where \mathbf{F}^e is the force vector and \mathbf{G}^e is the gravitational vector in ECEF coordinates.

Some of the derivatives in the ${}^i \ddot{\mathbf{r}}^e$ equation can be rewritten as:

$${}^e \ddot{\mathbf{r}}^e = {}^e \dot{\mathbf{V}}^e, \quad (8)$$

$${}^e \dot{\mathbf{r}}^e = \mathbf{V}^e. \quad (9)$$

If the above definitions are substituted in the ${}^i \ddot{\mathbf{r}}^e$ equation the following equation is obtained:

$$\frac{\mathbf{F}^e}{m} + \mathbf{G}^e = {}^e \dot{\mathbf{V}}^e + {}^e \dot{\omega}_{e/i}^e \times \mathbf{r}^e + 2\omega_{e/i}^e \times \mathbf{V}^e + \omega_{e/i}^e \times \omega_{e/i}^e \times \mathbf{r}^e. \quad (10)$$

Assuming the Earth's rotational velocity is constant gives:

$${}^e \dot{\omega}_{e/i}^e \times \mathbf{r}^e = 0. \quad (11)$$

Multiplying equation 10 by the direction cosine matrix $[\mathbf{C}]_{b/e}$ gives the following equation:

$$\frac{\mathbf{F}^b}{m} + \mathbf{G}^b = {}^e \dot{\mathbf{V}}^b + 2[\mathbf{C}]_{b/e} \omega_{e/i}^e \times \mathbf{V}^b + [\mathbf{C}]_{b/e} \omega_{e/i}^e \times \omega_{e/i}^e \times \mathbf{r}^e. \quad (12)$$

Rearranging for ${}^e \dot{\mathbf{V}}^b$ gives:

$${}^e \dot{\mathbf{V}}^b = \frac{\mathbf{F}^b}{m} - 2[\mathbf{C}]_{b/e} \omega_{e/i}^e \times \mathbf{V}^b + \mathbf{G}^b - [\mathbf{C}]_{b/e} \omega_{e/i}^e \times \omega_{e/i}^e \times \mathbf{r}^e, \quad (13)$$

where \mathbf{r}^e is given by:

$$\mathbf{r}^e = \begin{Bmatrix} (Re + h)\cos(\mu)\cos(l) \\ (Re + h)\cos(\mu)\sin(l) \\ (Re + h)\sin(\mu) \end{Bmatrix}. \quad (14)$$

Equation 13 gives the derivative of the velocity vector with respect to the EFEC coordinate system expressed in the body-fixed coordinate system. However, since the body-fixed coordinate system rotates with respect to the EFEC coordinate system the following equation must be used:

$${}^e \dot{\mathbf{V}}^b = {}^b \dot{\mathbf{V}}^b + \omega_{b/e}^b \times \mathbf{V}^b. \quad (15)$$

Substituting equation 15 into equation 13 gives:

$${}^b \dot{\mathbf{V}}^b = \frac{\mathbf{F}^b}{m} - \omega_{b/e}^b \times \mathbf{V}^b - 2[\mathbf{C}]_{b/e} \omega_{e/i}^e \times \mathbf{V}^b + \mathbf{G}^b - [\mathbf{C}]_{b/e} \omega_{e/i}^e \times \omega_{e/i}^e \times \mathbf{r}^e. \quad (16)$$

The above equation gives the derivative of the velocity vector with respect to the body-fixed coordinate system expressed in the body-fixed coordinate system. The flat-Earth equations of motion can be identified from equation 16. However, there are additional terms for the Coriolis and Centripetal acceleration due to the Earth's rotational velocity. Since angular velocities are additive, the angular

velocity of the body-fixed coordinate system relative to the ECI the coordinate system in body coordinates is given by:

$$\omega_{b/i}^b = \omega_{b/e}^b + [\mathbf{C}]_{b/e} \omega_{e/i}^e. \quad (17)$$

Rearranging the above equation for $\omega_{b/e}^b$ and substituting in the previous equation gives:

$$\begin{aligned} {}^b\dot{\mathbf{V}}^b = \frac{\mathbf{F}^b}{m} - (\omega_{b/i}^b - [\mathbf{C}]_{b/e} \omega_{e/i}^e) \times \mathbf{V}^b - 2[\mathbf{C}]_{b/e} \omega_{e/i}^e \times \mathbf{V}^b \\ + \mathbf{G}^b - [\mathbf{C}]_{b/e} \omega_{e/i}^e \times \omega_{e/i}^e \times \mathbf{r}^e. \end{aligned} \quad (18)$$

Further simplifying the above equation gives:

$${}^b\dot{\mathbf{V}}^b = \frac{\mathbf{F}^b}{m} - \omega_{b/i}^b \times \mathbf{V}^b - [\mathbf{C}]_{b/e} \omega_{e/i}^e \times \mathbf{V}^b + \mathbf{G}^b - [\mathbf{C}]_{b/e} \omega_{e/i}^e \times \omega_{e/i}^e \times \mathbf{r}^e. \quad (19)$$

The above are the translational aircraft equations of motion. The rotational aircraft equations of motion are given by:

$$\dot{\omega}_{b/i}^b = [\mathbf{I}]^{-1}(\mathbf{M}^b - \omega_{b/i}^b \times ([\mathbf{I}]\omega_{b/i}^b)), \quad (20)$$

and are similar to the flat-Earth rotational equations of motion with the difference that $\omega_{b/i}^b$ is the aircraft angular velocity vector with respect to the ECI coordinate system. The components of this angular velocity vector are not directly equal to p , q , and r which is the case for the flat-Earth equations of motion. Here they are given by $\omega_{b/f}^b$. In order to determine p , q and r , the rotation of the NED with respect to the EFEC coordinate system must be taken into account. Why is there rotation present between the NED and EFEC coordinate systems? Well, the NED coordinate system is always tangent to the Earth. If the aircraft starts its journey at $l = 0$ deg and ends its journey at $l = 90$ deg, the NED coordinate system would have rotated in order to stay tangent to the Earth. The angular velocity of the body-fixed to the ECEF coordinate system can be represented as:

$$\omega_{b/e}^b = \omega_{b/f}^b + [\mathbf{C}]_{b/f} \omega_{f/e}^f, \quad (21)$$

where $\omega_{b/f}^b$ is the angular velocity of the body-fixed with respect to the NED coordinate system and $\omega_{f/e}^f$ is the angular velocity of the NED with respect to the ECEF coordinate system.

The angular velocity of the NED coordinate system with respect to the ECEF coordinate system is given by:

$$\omega_{f/e}^f = \begin{Bmatrix} \dot{l}\cos(\mu) \\ -\dot{\mu} \\ -\dot{l}\sin(\mu) \end{Bmatrix}, \quad (22)$$

where:

$$\dot{\mu} = \frac{1}{Re + h} V_{f_x}, \quad (23)$$

$$\dot{l} = \frac{1}{Re + h} \cos(\mu) V_{f_y}, \quad (24)$$

$$\dot{h} = -V_{f_z}, \quad (25)$$

$$\mathbf{V}_f^f = [\mathbf{C}]_{f/b} \mathbf{V}_b^b. \quad (26)$$

Then the angular velocity of the body-fixed relative to the NED coordinate system is given by:

$$\omega_{b/f}^b = \omega_{b/e}^b - [\mathbf{C}]_{b/f} \omega_{f/e}^f, \quad (27)$$

which can be further simplified if using the previously derived definition for $\omega_{b/e}^b$:

$$\omega_{b/f}^b = \omega_{b/i}^b - [\mathbf{C}]_{b/f} \omega_{f/e}^f - [\mathbf{C}]_{b/e} \omega_{e/i}^e. \quad (28)$$

Summary

$$\mathbf{V}^b = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \quad (29)$$

$$\omega_{b/f}^b = \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} \quad (30)$$

$${}^b\dot{\mathbf{V}}^b = \begin{Bmatrix} \dot{u}_x \\ \dot{v}_y \\ \dot{w}_z \end{Bmatrix} = \frac{\mathbf{F}^b}{m} - \omega_{b/i}^b \times \mathbf{V}^b - [\mathbf{C}]_{b/e} \omega_{e/i}^e \times \mathbf{V}^b + \mathbf{G}^b - [\mathbf{C}]_{b/e} \omega_{e/i}^e \times \omega_{e/i}^e \times \mathbf{r}^e \quad (31)$$

$$\dot{\omega}_{b/i}^b = \begin{Bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{Bmatrix} = [\mathbf{I}]^{-1} (\mathbf{M}^b - \omega_{b/i}^b \times ([\mathbf{I}] \omega_{b/i}^b)) \quad (32)$$

$$\omega_{b/f}^b = \omega_{b/i}^b - [\mathbf{C}]_{b/f} \omega_{f/e}^f - [\mathbf{C}]_{b/e} \omega_{e/i}^e \quad (33)$$

$$\begin{aligned} [\dot{\mathbf{C}}]_{b/f} &= -[\boldsymbol{\Omega}_{b/f}] [\mathbf{C}]_{b/f} \\ [\dot{\mathbf{C}}]_{f/e} &= -[\boldsymbol{\Omega}_{f/e}] [\mathbf{C}]_{f/e} \end{aligned} \quad (34)$$

$$\begin{aligned} [\boldsymbol{\Omega}_{b/f}] &= \begin{bmatrix} 0 & -\omega_{z,b/f} & \omega_{y,b/f} \\ \omega_{z,b/f} & 0 & -\omega_{x,b/f} \\ -\omega_{y,b/f} & \omega_{x,b/f} & 0 \end{bmatrix} \\ [\boldsymbol{\Omega}_{f/e}] &= \begin{bmatrix} 0 & -\omega_{z,f/e} & \omega_{y,f/e} \\ \omega_{z,f/e} & 0 & -\omega_{x,f/e} \\ -\omega_{y,f/e} & \omega_{x,f/e} & 0 \end{bmatrix} \end{aligned} \quad (35)$$

$$\dot{\mathbf{x}} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \\ \dot{x}_9 \\ \dot{x}_{10} \\ \dot{x}_{11} \\ \vdots \\ \dot{x}_{18} \\ \dot{x}_{19} \\ \dot{x}_{20} \\ \vdots \\ \dot{x}_{27} \end{Bmatrix} = \begin{Bmatrix} \dot{u}_x \\ \dot{v}_y \\ \dot{w}_z \\ \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \\ \dot{\mu} \\ \dot{l} \\ \dot{h} \\ C\dot{11}_{f/e} \\ C\dot{12}_{f/e} \\ \vdots \\ C\dot{33}_{f/e} \\ C\dot{11}_{b/f} \\ C\dot{12}_{b/f} \\ \vdots \\ C\dot{33}_{b/f} \end{Bmatrix} \quad (36)$$